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# Advanced ODE-Lecture 10

## Lyapunov Stability for Linear Systems

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# Outline

- **Lyapunov Stability for Linear Autonomous Systems**
  - **Local Stability with Perturbation**
  - **Linearization**
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# Lyapunov Stability for Linear Autonomous Systems

The linear autonomous system is given by

$$x' = Ax. \quad (10.1)$$

If all eigenvalues of  $A$  satisfy  $\operatorname{Re} \lambda_j < 0$ ,  $A$  is said a **Hurwitz** matrix.

## 1) Lyapunov Method

Consider a **quadratic Lyapunov function** candidate

$$V(x) = x^T P x,$$

where  $P$  is a real symmetric positive definite matrix.

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The derivative of  $V$  along the trajectories of (10.1) is given by

$$V'(x) = x^T P x' + x'^T P x = x^T (PA + A^T P)x \stackrel{\Delta}{=} -x^T Q x,$$

where  $Q$  is a symmetric matrix defined by

$$PA + A^T P = -Q. \tag{10.2}$$

If  $Q$  is positive definite, we conclude by Theorem 9.1 that the origin is AS.

**Remark 10.1** (10.2) is called a Lyapunov equation of (10.1).

**Theorem 10.1**  $A$  is Hurwitz if and only if for any given positive definite symmetric matrix  $Q$ , there exists a positive definite symmetric matrix  $P$  that satisfies (10.2).

Moreover,  $P$  is unique for each given  $Q$  in (10.2).

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**Proof.** ( $\Leftarrow$ ) (Sufficiency) It is done by Theorem 9.1.

( $\Rightarrow$ ) (Necessity) If  $A$  is Hurwitz, define

$$P = \int_0^{\infty} \exp(A^T t) Q \exp(At) dt. \quad (10.3)$$

This integral (10.3) is well defined (convergent) because  $A$  is Hurwitz.  $P^T = P$  by definition. To show that  $P$  is positive definite, we use contradiction. If there were

$x \neq 0$  such that the quadratic form is  $x^T P x = 0$ . Then,

$$x^T P x = 0 \Rightarrow \int_0^{\infty} \exp(A^T t) Q \exp(At) dt = 0 \Rightarrow \exp(At)x \equiv 0, \quad \forall t \geq 0 \Rightarrow x = 0.$$

This contradiction shows that  $P$  is positive definite. Since

$$\begin{aligned} PA + A^T P &= \int_0^{\infty} \exp(A^T t) Q \exp(At) A dt + \int_0^{\infty} A^T \exp(A^T t) Q \exp(At) dt \\ &= \int_0^{\infty} \frac{d}{dt} \exp(A^T t) Q \exp(At) dt = \exp(A^T t) Q \exp(At) \Big|_0^{\infty} = -Q, \end{aligned}$$

which shows that  $P$  is a solution of (10.2). To show uniqueness, suppose there is another solution  $\tilde{P} \neq P$ . Then,

$$(P - \tilde{P})A + A^T(P - \tilde{P}) = 0.$$

Pre-multiplying by  $\exp(A^T t) \neq 0$  and post-multiplying by  $\exp(At) \neq 0$ , we obtain

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T(P - \tilde{P})]\exp(At) = \frac{d}{dt} \exp(A^T t)(P - \tilde{P})\exp(At).$$

Hence,

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv \text{Constant}, \quad \forall t \geq 0.$$

Since  $\exp(A^T t)(P - \tilde{P})\exp(At) \rightarrow 0$  as  $t \rightarrow \infty$ , then

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv 0.$$

Therefore,  $\tilde{P} = P$ .  $\square$

# Linear Autonomous System with Perturbation

Consider

$$x' = Ax + g(t, x), \quad (10.4)$$

where  $g(t, x)$  is continuous and locally Lipschitz in  $U$  containing the origin.

**Theorem 10.2 (Stability Theorem)** Let  $g(t, x)$  is continuous and locally Lipschitz in  $U$  containing the origin. If

$$\lim_{\|x\| \rightarrow 0} \frac{\|g(t, x)\|}{\|x\|} = 0,$$

holds uniformly in  $t$ , where  $t \geq t_0 \geq 0$  and  $A$  has all eigenvalues with negative real part, i.e.  $\operatorname{Re} \lambda_j(A) < 0$  for  $j = 1, 2, \dots, n$ , then  $x = 0$  of (10.4) is uniformly asymptotically stable.

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**Proof.** Since  $\lim_{\|x\| \rightarrow 0} \frac{\|g(t, x)\|}{\|x\|} = 0$  holds uniformly in  $t$ , there exists  $\varepsilon > 0$  for any given  $b > 0$  such that

$$\|g(t, x)\| \leq b \|x\| \quad \text{for all } t \geq 0,$$

provided  $\|x\| \leq \varepsilon$ . Then  $g(t, 0) \equiv 0$  for all  $t \geq 0$ , i.e.  $x = 0$  is equilibrium of (10.4). Since  $\operatorname{Re} \lambda_j(A) < 0$  for  $j = 1, 2, \dots, n$ , we can find  $K > 0$  and  $\mu > 0$  (Clue: using the formula of  $e^{At}$ ) s.t.

$$\|e^{A(t-t_0)}\| \leq K e^{-\mu(t-t_0)} \quad \text{for } t \geq t_0.$$

For  $\|x_0\| \leq \delta = \frac{\varepsilon}{K}$  and  $t_0 \geq 0$ , there exists a unique solution of (10.4), denoted by  $x(t, t_0, x_0)$  for  $t \in [t_0, \omega_+)$ . We show that  $\omega_+ = \infty$ . We can get the equivalent integral form as follows.

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$$x(t, t_0, x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-s)}g(s, x(s, t_0, x_0))ds, \quad t \in [t_0, \omega_+).$$

Then,

$$\|x(t, t_0, x_0)\| \leq Ke^{-\mu(t-t_0)}\|x_0\| + \int_{t_0}^t Ke^{-\mu(t-s)}\|g(s, x(s, t_0, x_0))\|ds, \quad t \in [t_0, \omega_+).$$

If  $\|x(t, t_0, x_0)\| \leq \varepsilon$  for all  $t \in [t_0, \omega_+)$ , then

$$\|g(t, x(t, t_0, x_0))\| \leq b\|x(t, t_0, x_0)\|, \quad t \in [t_0, \omega_+)$$

So we have

$$\|x(t, t_0, x_0)\| \leq Ke^{-\mu(t-t_0)}\|x_0\| + Kb \int_{t_0}^t e^{-\mu(t-s)}\|x(s, t_0, x_0)\|ds, \quad t \in [t_0, \omega_+).$$

Multiplying  $e^{\mu(t-t_0)}$  on both sides, we have

$$e^{\mu(t-t_0)}\|x(t, t_0, x_0)\| \leq K\|x_0\| + Kb \int_{t_0}^t e^{\mu(s-t_0)}\|x(s, t_0, x_0)\|ds, \quad t \in [t_0, \omega_+).$$

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Gronwall inequality yields

$$e^{\mu(t-t_0)} \|x(t, t_0, x_0)\| \leq K \|x_0\| e^{Kb(t-t_0)}, \quad t \in [t_0, \omega_+),$$

i.e.

$$\|x(t, t_0, x_0)\| \leq K \|x_0\| e^{-(\mu-Kb)(t-t_0)}, \quad t \in [t_0, \omega_+).$$

Since  $b > 0$  is arbitrary and only local result we concern, we choose  $b = \frac{\mu}{2K} > 0$ ,

and then we have

$$\|x(t, t_0, x_0)\| \leq K \|x_0\| e^{-\frac{\mu}{2}(t-t_0)} \leq \varepsilon e^{-\frac{\mu}{2}(t-t_0)} < \varepsilon, \quad t \in [t_0, \omega_+).$$

It follows that  $\omega_+ = \infty$  by Extensibility Theorem. Since  $\delta > 0$  and  $b > 0$  are independent of  $t_0 \geq 0$ ,  $x=0$  is uniformly asymptotically stable, in fact it is exponentially stable.  $\square$

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**Theorem 10.3 (Unstability Theorem)** Let  $g(t, x)$  is continuous and locally Lipschitz in  $U$  containing the origin. If

$$\lim_{\|x\| \rightarrow 0} \frac{\|g(t, x)\|}{\|x\|} = 0,$$

holds uniformly in  $t$ , where  $t \geq t_0 \geq 0$  and  $A$  has at least one eigenvalue with positive real part, i.e.  $\operatorname{Re} \lambda_{j_0}(A) > 0$ , then  $x = 0$  of (10.4) is unstable.

(Proof is omitted)

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# Linearization

Let us go back to the nonlinear system

$$x' = f(x), \quad (10.5)$$

where  $f : D \rightarrow \mathbb{R}^n$  is  $C^1$  and  $f(0) = 0$ . Then we write (10.5) by Taylor expansion as

$$f(x) = Ax + g(x), \quad (10.6)$$

where

$$A = Df(0), \text{ and } \lim_{\|x\| \rightarrow 0} \frac{\|g(x)\|}{\|x\|} = 0.$$

## Theorem 10.4 (Linearization)

1. The origin of (10.6) is AS if  $A$  is Herwitz.
2. The origin of (10.6) is unstable if  $\operatorname{Re} \lambda_j > 0$  for one or more of the eigenvalues of  $A$ .

**Proof.** Let  $A$  be a Hurwitz matrix. Then, by Theorem 10.1, for any positive definite symmetric matrix  $Q$ , the solution  $P$  of the Lyapunov equation (10.2) is positive definite. We take

$$V(x) = x^T P x$$

as a Lyapunov function candidate for (10.6). The derivative of  $V(x)$  along the trajectories of (10.6) is given by

$$\begin{aligned} V'(x) &= x^T P [Ax + g(x)] + [x^T A^T + g^T(x)] P x = x^T (P A + A^T P) x + 2x^T P g(x) \\ &= -x^T Q x + 2x^T P g(x). \end{aligned}$$

Since  $\lim_{\|x\| \rightarrow 0} \frac{\|g(x)\|}{\|x\|} = 0$ , there exists  $r > 0$  for any given  $b > 0$  such that

$$\|g(x)\| < b \|x\|, \quad \forall \|x\| < \varepsilon.$$

Hence,

$$V'(x) < -x^T Q x + 2b \|P\| \|x\|^2, \quad \forall \|x\| < \varepsilon,$$

but

$$-x^T Q x \geq \lambda_{\min}(Q) \|x\|^2$$

Note that  $\lambda_{\min}(Q)$  is real and positive since  $Q$  is symmetric and positive definite.

Thus

$$V'(x) < -[\lambda_{\min}(Q) - 2b \|P\|] \|x\|^2, \quad \forall \|x\| < \varepsilon.$$

Choosing  $b < \frac{\lambda_{\min}(Q)}{2\|P\|}$  ensures that  $V'(x)$  is negative definite. By Theorem 9.1, we

conclude that the origin of (10.6) is AS.

To prove the second part of the theorem, let us consider first the specific case when  $A$  has no eigenvalues on the imaginary axis. Then there exists an invertible matrix  $T$  such that

$$TAT^{-1} = \begin{pmatrix} -A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where  $A_1$  and  $A_2$  are Hurwitz matrices. The change of variables

$$z = Tx = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

transforms (10.6) into the form

$$\begin{cases} z_1' = -A_1 z_1 + g_1(z) \\ z_2' = -A_2 z_2 + g_2(z) \end{cases},$$

where  $g_j(z)$  satisfies

$$|g_j(z)| < b \|z\|, \quad \forall \|z\| \leq \varepsilon, \quad j = 1, 2.$$

Let  $Q_1$  and  $Q_2$  be positive definite symmetric matrices. Solving

$$P_j A_j + A_j^T P_j = -Q_j, \quad j = 1, 2,$$

yields a unique positive definite solutions  $P_1$  and  $P_2$ . Let

$$V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2 = z^T \begin{pmatrix} P_1 & 0 \\ 0 & -P_2 \end{pmatrix} z.$$

In the subspace  $z_2 = 0$ ,  $V(z) > 0$  at points arbitrarily close to the origin. Let

$$U = \{ z \in R^n \mid \|z\| \leq \varepsilon \text{ and } V(z) > 0 \}.$$



In  $U$ ,

$$\begin{aligned} V'(z) &= -z_1^T(P_1A_1 + A_1^TP_1)z_1 + 2z_1^TP_1g_1(z) - z_2^T(P_2A_2 + A_2^TP_2)z_2 - 2z_2^TP_2g_2(z) \\ &= z_1^TQ_1z_1 + z_2^TQ_2z_2 + 2z^T \begin{pmatrix} P_1g_1(z) \\ -P_2g_2(z) \end{pmatrix} \\ &\geq \lambda_{\min}(Q_1)\|z_1\|^2 + \lambda_{\min}(Q_2)\|z_2\|^2 - 2\|z\| \sqrt{\|P_1\|^2\|g_1(z)\|^2 + \|P_2\|^2\|g_2(z)\|^2} \\ &> (\alpha - 2\sqrt{2}\beta b)\|z\|^2, \end{aligned}$$

where

$$\alpha = \min\{\lambda_{\min}(Q_1), \lambda_{\min}(Q_2)\}. \quad \beta = \max\{\|P_1\|, \|P_2\|\}.$$

Thus, choosing  $b < \frac{\alpha}{2\sqrt{2}\beta}$  ensures that  $V'(z) > 0$  in  $U$ . Therefore, by Theorem

9.3, the origin of (10.6) is unstable.

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Let us study now the general case when  $A$  may have eigenvalues on the imaginary axis meanwhile  $A$  has eigenvalues with positive real parts. By a simple trick of shifting the imaginary axis, we suppose that  $A$  has  $m$  eigenvalues with  $\operatorname{Re} \lambda_j > \delta > 0$ . Then, the matrix  $A - \frac{\delta}{2}I$  has  $m$  eigenvalues in the open right-half plane, but no eigenvalues on the imaginary axis. Then, there exist  $P = P^T$  and  $Q = Q^T$  such that

$$P\left(A - \frac{\delta}{2}I\right) + \left(A - \frac{\delta}{2}I\right)^T P = Q,$$

where  $V(x) = x^T P x > 0$  at points arbitrarily close to the origin. The derivative of  $V(x)$  along the trajectories of (10.6) is given by

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$$\begin{aligned}
V'(x) &= x^T (PA + A^T p)x + 2x^T P g(x) \\
&= x^T \left[ P \left( A - \frac{\delta}{2} I \right) + \left( A - \frac{\delta}{2} I \right)^T P \right] x + \delta x^T P x + 2x^T P g(x) \\
&= x^T Q x + \delta V(x) + 2x^T P g(x).
\end{aligned}$$

In the set

$$U = \{x \in R^n \mid \|x\| \leq \varepsilon \text{ and } V(x) > 0\},$$

where  $\varepsilon$  is chosen such that  $\|g(x)\| \leq b \|x\|$  for  $\|x\| < \varepsilon$ ,  $V'(x)$  satisfies

$$V'(x) \geq \lambda_{\min}(Q) \|x\|^2 - 2 \|P\| \|x\| \|g(x)\| \geq (\lambda_{\min}(Q) - 2b \|P\|) \|x\|^2,$$

which is positive definite if  $b < \frac{\lambda_{\min}(Q)}{2 \|P\|}$ . The origin of (10.6) is unstable.  $\square$

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**Remark 10.2** Theorem 10.3 does not say anything about the case when  $\operatorname{Re} \lambda_j \leq 0$  for all  $j$ , with  $\operatorname{Re} \lambda_j = 0$  for some  $j$ . In this case, linearization fails to determine stability of equilibrium. Center manifold theory may apply.

**Example 10.1** For  $x' = ax^3$ , where  $a$  is a parameter. Linearization yields

$$A = Df(0) = 3ax^2 \big|_{x=0} = 0.$$

Hence, linearization fails. This failure is essential in the sense that  $x = 0$  could be AS, stable, or unstable, depending on the value of the parameter  $a$ .

Take  $V(x) = x^4 > 0$  as a Lyapunov function.  $V'(x) = 4ax^6$ . Then,

If  $a < 0$ ,  $V'(x) < 0 \Rightarrow x = 0$  is AS. If  $a = 0$ ,  $V'(x) \equiv 0 \Rightarrow x = 0$  is stable. If

$a > 0$ ,  $V'(x) > 0 \Rightarrow x = 0$  is unstable.

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**Example 10.2** Consider

$$x_1' = -x_2 - x_2(x_1^2 + x_2^2); \quad x_2' = x_1 + x_1(x_1^2 + x_2^2),$$

where  $A = Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  has  $\lambda = \pm i$  with  $\operatorname{Re} \lambda(A) = 0$ . Since  $x = 0$  is a center

of  $x' = Df(0)x$ , it is stable but not asymptotically stable.

Introducing the polar coordinate transformation

$$x_1 = r \cos \theta; \quad x_2 = r \sin \theta,$$

we have  $(x_1 \frac{dx_1}{dt} + x_2 \frac{dx_2}{dt} = r \frac{dr}{dt}; \quad x_1 \frac{dx_2}{dt} - x_2 \frac{dx_1}{dt} = r^2 \frac{d\theta}{dt}$ ; The detail leaves for students)

$$\frac{dr}{dt} = 0; \quad \frac{d\theta}{dt} = 1 + r^2.$$

Solving the equations yields the solution:  $r(t) = r_0^2$ . So  $x = 0$  is still stable ( $x = 0$  is a center of the original equations).

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# Summary

1. Lyapunov stability for linear autonomous systems has two methods both by an eigenvalue approach and Lyapunov matrix approach.
  2. Linearization works for hyperbolic type of equilibrium and for local results. Is it possible for global one? It is no in general! However it is yes conditional! It needs further study.
  3. Lyapunov Matrix equation gives a way to find a Lyapunov function. How about for general case? It is refers to Lyapunov converse theorem.
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## Homework

- Review today's contents of class.



